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The material force acting on a screw dislocation in the presence of a multi-layered circular inclusion

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Abstract

In this paper, we study the interaction of a screw dislocation with a multi-layered interphase between a circularly cylindrical inclusion and a matrix. The layers are coaxial cylinders of annular cross-sections with arbitrary radii and different shear moduli. The number of layers may also be arbitrary. Continuity of traction and displacement across all interfaces is assumed. We extend Honein et al.'s solution of circularly cylindrical layered media in anti-plane elastostatics to the case where all the singularities reside inside the inclusion core. The solution to this heterogeneous problem is given explicitly, for arbitrary singularities, as a rapidly convergent Laurent series, whose coefficients are expressed in terms of those of the complex potential of a corresponding homogeneous problem with the same singularities. We then consider the two particular cases of a screw dislocation, where, in the first instance, the dislocation resides inside the matrix, while, in the second instance, it is located in the inclusion core. In both instances, the Peach–Koehler force acting on the dislocation is calculated explicitly as a rapidly convergent series. We present several examples, where the effect of the layers on the material force is examined.

Keywords: Material force; Peach–Koehler; Screw dislocation; Multi-layered; Inhomogeneity; Heterogenization; Anti-plane elasticity; Circularly cylindrical; Layered media

1. Introduction

In a recent paper, Liu et al. (2003) have considered the problem of the interaction of a screw dislocation with an interphase layer between a circular inclusion and a matrix. They have combined the sectionally

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holomorphic function, Cauchy integral and Laurent series expansion techniques to arrive at explicit series solutions for the two cases when the screw dislocation is located either in the inclusion or in the matrix. It turns out that the latter case is a particular instance of a much more general treatment carried out some years earlier by Honein et al. (1994). These authors have treated the case where an *arbitrary* number of layers are embedded between the inclusion and the matrix, which may be subjected to *arbitrary* singularities and/or loading. Their method, a procedure which they have termed "heterogenization", consists in relating the solution of this heterogeneous problem to that of a corresponding homogeneous one, where the inclusion and layers are absent and the matrix material occupies the whole space. Furthermore, they have shown that, once the known solution to the homogeneous problem is expanded in a power series, the solution to the heterogeneous problem can then be written down immediately as a rapidly convergent series. They have also demonstrated that the transformation effecting this result can be expressed concisely and elegantly in terms of a group structure on the set $I = (-1, 1)$, of real numbers, x, such that $-1 < x < 1$. The problem of a screw dislocation residing inside a matrix in the presence of an arbitrary number of layers was thus solved; since it was provided in that work as a specific example, which Liu et al. have failed to cite.

The purpose of this paper is twofold. First, we extend Honein et al's result to the case where the singularities are all inside the inclusion. Second, as a particular case, we study in detail the interaction of a screw dislocation with a multi-layered interphase between a circular inclusion and a matrix. Explicit rapidly convergent Laurent series solutions will be provided for the two cases when the screw dislocation is located either in the matrix or in the inclusion. The material (or Peach–Koehler) force on the screw dislocation will be evaluated using Budiansky and Rice's formula (1973) for the J-integral in anti-plane elastostatics. In the case of a screw dislocation residing in the matrix, and apart from a missing factor due to a misprint, our result for the Peach–Koehler force agrees with the one obtained by Honein et al. (1994).

In the particular case, when the number of layers is reduced to just one, our derivation yields the same results as the ones obtained by Liu et al. (2003). However, our formulas are expressed in a more compact form, especially for the case where the dislocation is inside the inclusion.

Dislocations behavior may explain certain strengthening or hardening mechanism in a number of traditional and composite materials (Hirth and Lothe, 1982). The interaction of dislocations with inclusions (inhomogeneities) has received some considerable attention from the research community. For instance, Gong and Meguid (1994) consider the interaction effects between a screw dislocation and an elastic elliptical inhomogeneity. A general solution to the problem is obtained through the use of conformal mapping and the complex variable method. For earlier work, the reader may wish to consult, for example, Smith (1968) and Sendeckyj (1970). See also the references cited in Liu et al. (2003). Additionally, dislocations can be used to model cracks. Therefore our solution can be used as a building block to model cracks in multi-layered media, see for example Chao and Young (1998) and the references therein.

The remainder of this paper is organized as follows. After briefly recollecting the basic notation of the complex formulation of antiplane elastostatics, we proceed to recall the solution to a multi-layered circular inclusion perfectly bonded to an infinite matrix. This elastic system may be subjected to loading or singularities, which produce an anti-plane deformation but are otherwise arbitrary. In this instance the singularities are assumed to reside in the matrix.

Then we consider the particular case of a screw dislocation residing inside the matrix. We write down the solution explicitly in the form of a rapidly convergent power series and we calculate the material force acting on the screw dislocation in the presence of an arbitrary number of layers.

Next, we turn our attention to the case where the singularities are inside the innermost region of the inclusion (the inclusion core). Following the same line of reasoning as presented by Honein et al. (1994), we express the solution to this heterogeneous problem in terms of the solution of the corresponding homogeneous problem, i.e., when the layers and the matrix are absent and the inclusion, still subjected to the same singularities, occupies the whole space. We show that this can be achieved by exploiting a connection between the solution to the heterogeneous problem and a group structure on the set $I = (-1, 1)$, of real numbers x, such that $-1 < x < 1$. It is the same group structure that served to write down the solution when the singularities were in the matrix.

The solution is then worked out for the case when a screw dislocation resides in the innermost region of the inclusion. Again, the solution is derived in the form of a rapidly convergent Laurent series expansion and the material force acting on the screw dislocation is expediently calculated, using the residue theorem and a formula due to Budiansky and Rice (1973), relating the J-integral to the complex potential in anti-plane elastostatics. Finally, we examine in detail the influence of the layers material parameters and thickness on that force and we draw some conclusions.

2. Complex formulation of anti-plane elastostatics

Under anti-plane deformation, the displacement field satisfies

$$u_1 = u_2 \equiv 0, \quad u = u_3 = u_3(x_1, x_2), \tag{1}$$

i.e., the only nonvanishing component of displacement, with respect to a Cartesian coordinate system $Ox_1x_2x_3$, is $u = u_3$ which is a function of the coordinates x_1 and x_2 only.

As is well known, the displacement field u, for a homogeneous material, can be expressed, in the case of antiplane elastostatics, in terms of an analytic complex function ϕ of a complex variable $z = x_1 + \mathrm{i}x_2$, namely

$$u = \frac{1}{\mu}\Im\{\phi(z)\}, \tag{2}$$

where μ is the shear modulus and $\Im$ stands for the imaginary part of the argument. Then the nonvanishing stress components, in Cartesian coordinates, are related to ϕ by

$$\sigma_{23} + \mathrm{i}\sigma_{13} = \phi' \tag{3}$$

and, in polar coordinates, by

$$\sigma_{\theta 3} + \mathrm{i}\sigma_{r3} = \mathrm{e}^{\mathrm{i}\theta}\phi', \tag{4}$$

where, throughout this paper, prime indicates differentiation with respect to the complex variable z.

2.1. The structure of the complex potential

Consider a stress distribution in a domain, Ω, specified by the complex potential, ϕ, see Fig. 1.

Eq. (3) demands that ϕ' be a single-valued function in Ω. It follows that ϕ itself must be a single-valued function if Ω is simply connected and free of singularities.

We consider now the case where Ω is a region that contains singularities. To this end, we suppose that u is singular at a point z_k and we further assume that u increases by b_k, where b_k is a real number, as we traverse in the counterclockwise direction along a contour enclosing z_k. The quantity thus introduced, b_k is the third component of the Burgers vector defining a screw dislocation. Thus, for m such singularities, the complex potential ϕ can be written in the form

$$\phi = \sum_{k=1}^{m}\left(\frac{\mu b_k}{2\pi} + \mathrm{i}\frac{t_k}{2\pi}\right)\log(z - z_k) + \phi^*, \tag{5}$$

where ϕ^* is holomorphic and single-valued in Ω and t_k is a real number which will be defined shortly.

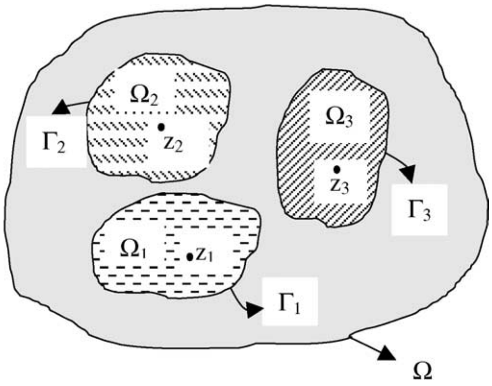

Fig. 1. An elastic field in a domain, Ω, with possible singularities at the points: z_1, z_2, ...

In the neighborhood of z_k, we have

$$\phi = \left(\frac{\mu b_k}{2\pi} + \mathrm{i}\frac{t_k}{2\pi}\right)\log r + \mathrm{i}\theta + \text{a holomorphic function,} \tag{6}$$

where $r = |z - z_k|$ and $\theta = \arg(z - z_k)$.

Hence, it follows from Eqs. (2) and (3) that the displacement increases by b_k and the traction by t_k as one traverses along a contour surrounding the singularity located at z_k in the counterclockwise direction. Thus, t_k represents the total traction around a path enclosing the singularity.

The form (6) holds also for a multiply connected region with m contours Γ_k, z_k being a point inside the region Ω_k, even when Ω is an infinite region, provided we require that ϕ^* be analytic at the point at infinity.

3. Solution of a multi-layered inclusion: all singularities are in the matrix

In this section, we consider the system shown in Fig. 2 whose cross-section consists of $n - 1$ concentric circular rings $D_1, \ldots, D_{n-1}$ perfectly bonded along their common boundaries. The innermost boundary is

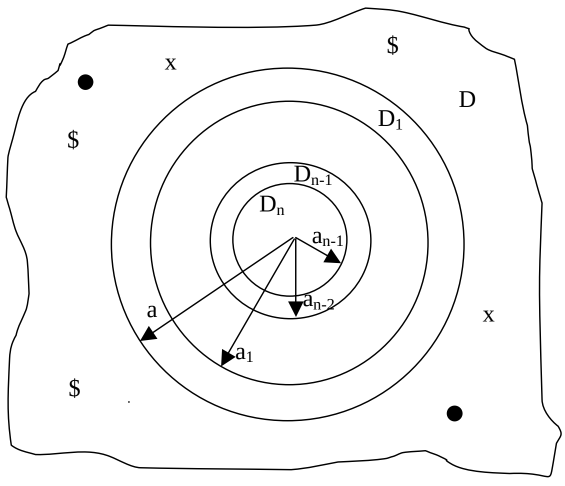

Fig. 2. A multi-layered domain with all singularities in the matrix.

perfectly bonded to a circular domain D_n, while the outermost boundary, a circle of radius $a = a_0$, is perfectly bonded to a matrix D, of infinite extent, occupied by a material of shear modulus $\mu = \mu_0$ and subjected to arbitrary loading/singularities. The domain D_i is occupied by a material of shear modulus μ_i, $i = 1,\ldots,n$, the outer radius of D_i is a_{i-1}, $i = 1,\ldots,n$, while its inner radius is a_i, $i = 1,\ldots,n-1$.

We designate by ϕ the solution to the corresponding homogeneous problem, i.e., the problem when the multi-layered inclusion is absent and the domain D, still subjected to the same loading/singularities, occupies the whole space.

We seek to express the solution to the heterogeneous problem in terms of ϕ. Thus we set

$$\text{in } D : \Phi = \phi + H_a(f), \tag{7}$$

$$\text{in } D_i : \Phi_i = \phi + \phi_i + H_{a_i}(f_i), \quad i = 1,\ldots,n-1, \tag{8}$$

$$\text{in } D_n : \Phi_n = \phi + \phi_n. \tag{9}$$

The functions ϕ_1 and f are analytic in the disk $r < a$, while ϕ_{i+1} and f_i are analytic in the disk $r < a_i$, for $i = 1,\ldots,n-1$.

Here H_{a_i} is the "hat" transformation, which is defined by $H_{a_i}(f_i) = \overline{f_i(a_i^2/\bar{z})}$, where the overbar denotes complex conjugation.

Upon using the notation

$$m_i = \mu_i/\mu_{i-1}, \quad i = 1,\ldots,n, \tag{10}$$

$$q_i = a_i^2/a_{i-1}^2, \quad i = 1,\ldots,n-1 \tag{11}$$

and

$$\alpha_i = (m_i - 1)(m_i + 1)^{-1}, \quad i = 1,\ldots,n, \tag{12}$$

it can then be shown (see Honein et al., 1994) that, if the solution to the homogeneous problem is expressed by a Taylor series expansion as $\phi = \sum_{k=1}^{\infty} b_k z^k$, where b_k, k=1, ..., ∞ are complex coefficients, then the solution to the heterogeneous problem can be written down as

$$f = \sum_{k=1}^{\infty} \alpha_1 * q_1^k(\alpha_2 * q_2^k(\cdots)) b_k z^k, \tag{13}$$

where the $*$ operation is defined by

$$x * y = \frac{x+y}{1+xy}. \tag{14}$$

The functions ϕ_i and f_i, which give the solution inside the fiber, can be obtained by induction according to

$$2\phi_{i+1} = (m_{i+1} - 1)\phi + (m_{i+1} + 1)\phi_i + (1 - m_{i+1}) f_i, \quad i = 0,\ldots,n-1 \tag{15}$$

and

$$2Q_{i+1} f_{i+1} = (1 - m_{i+1})\phi + (1 - m_{i+1})\phi_i + (m_{i+1} + 1) f_i, \quad i = 0,\ldots,n-2 \tag{16}$$

with the convention $\phi_0 = 0$ and $f_0 = f$.

These last relations have been obtained by enforcing the continuity of traction and displacement across the boundary common to D_i and D_{i+1}. The Q_j operator on an analytic function h is defined by

$$Q_j h(z) = h(q_j z), \quad j = 1,\ldots,n-1. \tag{17}$$

The operation $*$ endows the set $I = (-1, 1)$ of real numbers x such that $-1 < x < 1$ with a group structure.

Eq. (13) provides an explicit solution of the heterogeneous problem, with an arbitrary number of layers, in terms of that of the corresponding homogeneous one.

If the matrix is subjected to remote uniform shear loading then the complex potential for the homogeneous problem is given by $\phi = \tau z$, where τ is a complex constant. The solution of the multilayered fiber problem can thus be written down immediately as

$$f = \tau\alpha_1 * q_1(\alpha_2 * q_2(\cdots))z, \tag{18}$$

which shows that the multi-layered inclusion may be replaced by a single homogeneous inclusion of equivalent relative modulus given by

$$\alpha_{\text{eq}}^{(1)} = \alpha_1 * q_1(\alpha_2 * q_2(\cdots)) \tag{19}$$

without affecting the state of stress in the matrix. However, the stress distribution inside the inclusion will depend on the arrangement of the layers and on their material properties.

More generally, we may call

$$\alpha_{\text{eq}}^{(k)} = \alpha_1 * q_1^k(\alpha_2 * q_2^k(\cdots)), \tag{20}$$

the equivalent relative modulus of order k, so that the solution in the matrix to the heterogeneous problem can be expressed as

$$f = \sum_{k=1}^{\infty} \alpha_{\text{eq}}^{(k)} b_k z^k, \tag{21}$$

provided that the solution to the corresponding homogeneous problem is given by the complex potential $\phi = \sum_{k=1}^{\infty} b_k z^k$.

4. The material force on a screw dislocation residing in the matrix

In this section, we consider the example of a screw dislocation interacting with a multilayered circular inclusion.

Without loss of generality, we assume that the screw dislocation is located at the point, $-c$, of the x_1-axis where $|c| > a$ (see Fig. 3).

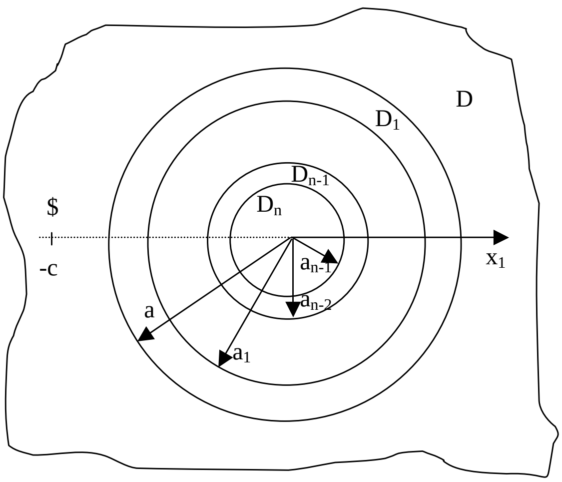

Fig. 3. A screw dislocation in the matrix interacting with a multi-layered inclusion.

The complex potential of the corresponding homogeneous problem (i.e., when the multi-layered inclusion is absent and the matrix material occupies the whole space) is given by (see Eq. (5))

$$\phi = \frac{b\mu}{2\pi}\ln(z + c), \tag{22}$$

where b is a real constant designating the third component of the Burgers vector.

In the domain $r < a$, ϕ has the expansion (up to an additive constant, which can be discarded)

$$\phi = \frac{b\mu}{2\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{z^k}{kc^k}. \tag{23}$$

Thus, it immediately follows from Eq. (21) that:

$$f = \frac{b\mu}{2\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\alpha_{\mathrm{eq}}^{(k)}\frac{z^k}{kc^k}. \tag{24}$$

Once we have an explicit solution in terms of the complex potential, the simplest and most expedient way to evaluate the material (or Peach–Koehler) force acting on a screw dislocation is through a formula for the J_1 and J_2 integrals derived by Budiansky and Rice (1973). In general, these two integrals represent the material force, or the negative of the energy release rate, as a defect enclosed by the contour of integration undergoes a unit displacement in the x_1 and the x_2 direction, respectively. In anti-plane elastostatics, the formula can be written as

$$J_1 - \mathrm{i}J_2 = -\frac{\mathrm{i}}{2\mu}\oint_C (\Phi')^2\,\mathrm{d}z, \tag{25}$$

where Φ is the complex potential and C is a contour surrounding the defect (the screw dislocation, in our case).

Upon substituting (7) into (25) and noting that ϕ is given by (22), it can easily be shown, by applying the residue theorem, that the above-mentioned equation takes the form

$$J_1 - iJ_2 = b(H(f))'\big|_{z=-c}, \tag{26}$$

which, upon using (24), leads to

$$J_1 - \mathrm{i}J_2 = -\frac{\mu b^2}{2\pi c}\sum_{k=1}^{\infty}\alpha_{\mathrm{eq}}^{(k)}\left(\frac{a}{c}\right)^{2k}. \tag{27}$$

This result yields the material force (or the Peach–Koehler force, in the case of a screw dislocation) acting on the point singularity.

The series given by (27) is rapidly convergent for it has a convergent geometric series as a majorant; since $|\alpha_{\mathrm{eq}}^{(k)}| = |\alpha_1 * q_1^k(\alpha_2 * q_2^k(\cdots))| < 1$. Furthermore, from this last observation an upper bound on the error when the series (27) is truncated for the purpose of numerical computation can be easily derived.

The equations derived in this section agree with the ones obtained by Honein et al. (1994). However, their final expression for the Peach–Koehler force contains some misprints.

For the case $n = 2$ (a homogeneous interphase layer between the inclusion and the matrix), our result for the complex function f reduces to

$$f = \frac{b\mu}{2\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\alpha_1 + \alpha_2 q_1^k}{1 + \alpha_1\alpha_2 q_1^k}\frac{z^k}{kc^k} \tag{28}$$

and Eq. (27) takes the form

$$J_1 - iJ_2 = -\frac{\mu b^2}{2\pi c}\sum_{k=1}^{\infty}\frac{\alpha_1 + \alpha_2 q_1^k}{1 + \alpha_1\alpha_2 q_1^k}\left(\frac{a}{c}\right)^{2k}. \tag{29}$$

These last two equations are in agreement with the ones arrived at by Liu et al. (2003), although our formulas are written down in a more compact form.

The case of a multi-layered interphase around a hole can be obtained from Eq. (27) by setting

$$\alpha_n = -1, \tag{30}$$

while that of a multi-layered interphase surrounding a rigid inclusion is derived by putting

$$\alpha_n = 1. \tag{31}$$

Many other particular cases in the literature can be derived from (27) as well by selecting appropriate values for the parameters $\alpha_1, \alpha_2, \ldots$ and $q_1, q_2, \ldots$, which are the only relevant ones, as can be easily seen upon inspection of Eq. (20).

It is well known that a circular hole attracts a dislocation. The presence of a single layer, which is softer than the matrix, does not alter this statement. However equilibrium positions exist when a circular hole is coated with a material which is harder than the matrix. Fig. 4 illustrates this point.

In Fig. 4, we plot the dimensionless material force $F_0 = \frac{2\pi a}{\mu b^2}J_1$, as a function of the dimensionless dislocation location given by $\frac{c}{a}$, for various values of the shear modulus of a homogeneous single layer coating the hole rim. As can be seen, for small values of μ_1 (e.g., 0.5), the effect of the hole remains dominant even as the dislocation approaches the boundary. However as the value of μ_1 increases the layer starts asserting

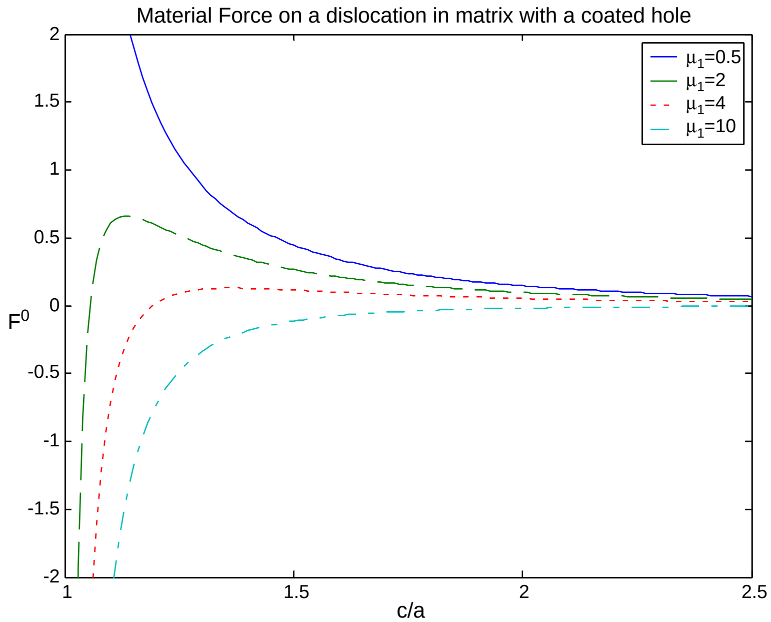

Fig. 4. The material force acting on a screw dislocation in a matrix due to the presence of a hole coated with a single homogeneous layer for various values of the shear modulus μ_1. The other parameters are $\mu_0 = 1$, $a_0 = 1$ and $a_1 = 0.9$.

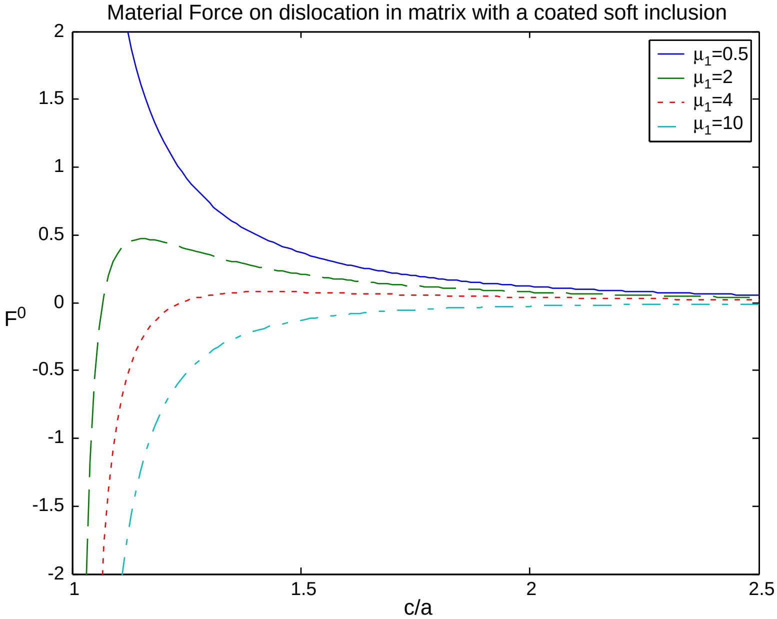

Fig. 5. The material force acting on a screw dislocation in a matrix due to the presence of a soft inclusion coated with a single homogeneous layer for various values of the shear modulus μ_1. The other parameters are $\mu_0 = 1$, μ_2=0.1, $a_0 = 1$ and $a_1 = 0.9$.

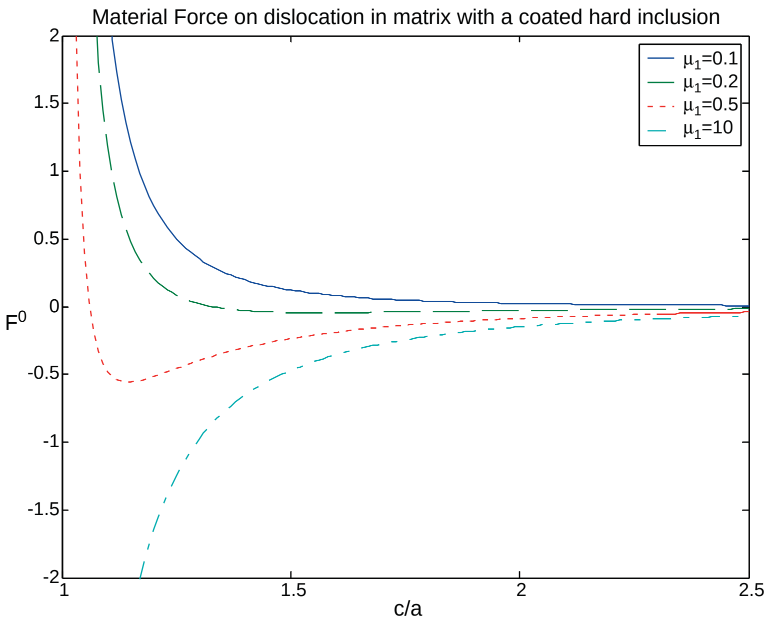

Fig. 6. The material force acting on a screw dislocation in a matrix due to the presence of a hard inclusion coated with a single homogeneous layer for various values of the shear modulus μ_1. The other parameters are $\mu_0 = 1$, $\mu_2 = 20$, $a_0 = 1$ and $a_1 = 0.9$.

itself and the force which is attractive away from the hole becomes repulsive as the dislocation nears the boundary, thus giving rise to a stable equilibrium position. As the value of the shear modulus increases

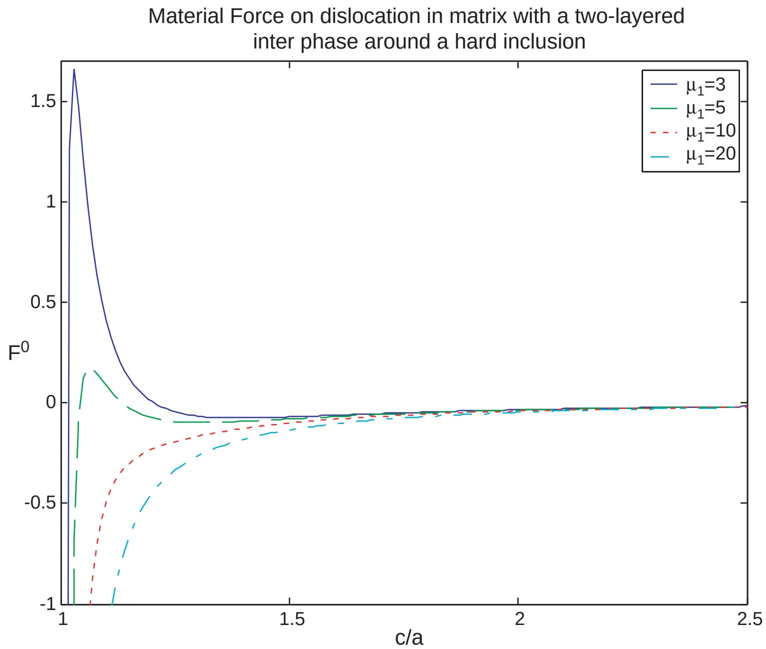

Fig. 7. The material force acting on a screw dislocation in a matrix due to the presence of a hard inclusion coated with two homogeneous layers for various values of the shear modulus μ_1. The other parameters are $\mu_0 = 1$, $\mu_2 = 0.08$, $\mu_3 = 5$, $a_0 = 1$, $a_1 = 0.99$ and a_2=0.96.

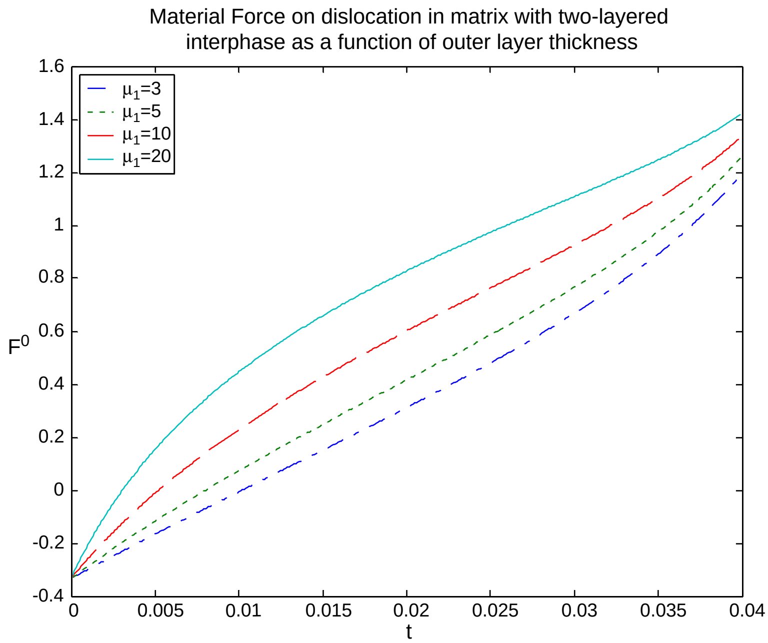

Fig. 8. The material force, plotted as a function of layer thickness, acting on a screw dislocation in a matrix due to the presence of a hard inclusion coated with two homogeneous layers for various values of the shear modulus μ_1. The parameters are $\mu_0 = 1$, $\mu_2 = 0.08$, $\mu_3 = 5$, $a_0 = 1$, $a_1 = a_0 - t$, $a_2 = 0.96$ and $c = 1.2$.

further the layer becomes dominant and the force becomes repulsive throughout the range and decays rapidly to zero as we move away from the hole.

The character of the solution is not altered by much if the hole is replaced by a soft inclusion. This can be clearly shown upon inspecting Fig. 5.

When the soft inclusion is replaced with a hard one, the sign of the force is reversed as shown in Fig. 6 and the stable equilibrium positions become unstable.

As shown in Fig. 7, the presence of an additional layer will alter the character of the solution and for certain combinations of the shear moduli, two equilibrium positions are possible. However, only one of them would be stable. Also, this figure shows that for a given thickness, any of the layers may become dominant if its shear modulus is high enough.

Fig. 8 shows the variation of the material force, on a screw dislocation with a fixed location, as a function of the thickness of one of the layers. Typically, the layer becomes dominant as its thickness increases and the figure shows the typical rise in the force magnitude with the increase in thickness.

5. Solution of a multi-layered inclusion: all singularities are in the inclusion core

In this section, we consider the same system shown in Fig. 2. However, this time we assume that all singularities reside in the inclusion core; that is in the domain D_n, see Fig. 9.

We designate by ϕ the solution to the corresponding homogeneous problem, i.e., the problem when the layers and the matrix are absent while the whole space is occupied by the same material as that of the inclusion core, which is still subjected to the same singularities.

As pointed out in Section 2.1, the complex potential ϕ can be written down as

$$\phi = A\ln(z) + \phi_0, \tag{32}$$

where A is a complex constant and ϕ_0 is analytic in the domain defined by $|r| > a_{n-1}$, *including the point at infinity*.

Our objective is to express the solution to the heterogeneous problem in terms of ϕ. In view of the principle of superposition, we can treat the terms $A\ln(z)$ and ϕ_0 separately. Furthermore, when only the first term is present, we can divide the problem into two separate cases: the first being that of a screw dislocation

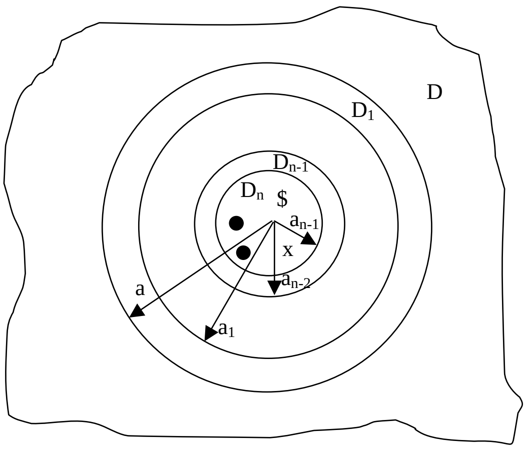

Fig. 9. A multi-layered elastic system; all singularities are in the inclusion core.

at the origin (A is purely real) and the second being that of a point force at the origin (A is purely imaginary). In what follows, these three cases will be considered independently.

5.1. Case 1: screw dislocation at the origin

In this subsection, we shall consider a screw dislocation acting at the origin. As is well known the complex potential of the corresponding homogeneous problem is given by the following expression.

$$\phi = \frac{b\mu_n}{2\pi}\ln(z), \tag{33}$$

where b, a real number, designates the third component of the Burgers vector. It can be easily seen that the solution to the heterogeneous problem may be written down immediately as

$$\text{in } D_n: \ \Phi_n = \frac{b\mu_n}{2\pi}\ln(z), \tag{34}$$

$$\text{in } D_{n-1}: \ \Phi_{n-1} = \frac{b\mu_{n-1}}{2\pi}\ln(z), \tag{35}$$

$$\vdots$$

$$\text{in } D_1: \ \Phi_1 = \frac{b\mu_1}{2\pi}\ln(z), \tag{36}$$

$$\text{in } D: \ \Phi = \frac{b\mu}{2\pi}\ln(z). \tag{37}$$

The above-mentioned solution ensures the continuity of displacement and traction at all interfaces.

5.2. Case 2: point force at the origin

When a point force is located at the origin, the complex potential for the corresponding homogeneous problem is given by

$$\phi = \mathrm{i}\mathrm{t}\ln(z), \tag{38}$$

where t is a real constant.

We seek the solution to the heterogeneous problem in the form.

$$\text{in } D_n: \ \Phi_n = \mathrm{i}\mathrm{t}\ln(z), \tag{39}$$

$$\text{in } D_{n-1}: \ \Phi_{n-1} = \mathrm{i}\mathrm{t}\ln(z) + \mathrm{i}c_{n-1}, \tag{40}$$

$$\vdots$$

$$\text{in } D_1: \ \Phi_1 = \mathrm{i}\mathrm{t}\ln(z) + \mathrm{i}c_1, \tag{41}$$

$$\text{in } D: \ \Phi = \mathrm{i}\mathrm{t}\ln(z) + \mathrm{i}c_0, \tag{42}$$

where $c_0, c_1, \ldots, c_{n-1}$ are real constants.

By imposing the continuity of displacement and traction along the common boundaries, it can be shown that the constants, c_k, can be obtained using the following recursive formulas:

$$c_{n-1} = (m_n^{-1} - 1)\mathrm{t}\ln(a_n - 1), \tag{43}$$

$$c_j = (m_{j+1}^{-1} - 1)\mathrm{t}\ln(a_j) + m_{j+1}^{-1}c_{j+1}, \quad j = n-2, \ldots, 0, \tag{44}$$

where we recall that m_j, $j = 1, \ldots, n$ have been defined by Eq. (10), with the convention that $\mu_0 = \mu$.

5.3. Case 3: ϕ is analytic in the domain $r > a_{n-1}$ including the point at infinity

In this section, we consider the system shown in Fig. 9. We assume, however, that the singularities, all located in the domain D_n, give rise in the corresponding homogeneous problem to a complex potential ϕ that is analytic in the domain $r > a_{n-1}$, *including the point at infinity*.

We seek the solution to the heterogeneous problem in the form

$$\text{in } D_n: \ \Phi_n = \phi + H_{a_{n-1}}(f_n), \tag{45}$$

$$\text{in } D_i: \ \Phi_i = \phi + \phi_i + H_{a_{i-1}}(f_i), \quad i = n-1, \ldots, 1, \tag{46}$$

$$\text{in } D: \ \Phi = \phi + \phi_0. \tag{47}$$

The functions ϕ_0 and f_1 are analytic in the domain $r > a$, *including the point at infinity*, while ϕ_i and f_{i+1} are analytic in the domain $r > a_i$, *including the point at infinity*, for $i = n-1, \ldots, 1$.

For later convenience, we define

$$p_j = q_j^{-1} = \left(\frac{a_{j-1}}{a_j}\right)^2, \quad j = 1, \ldots, n-1, \tag{48}$$

where we set $a_0 = a$.

Following the line of reasoning outlined in Honein et al. (1994), let R_n be the image operator associated with $D_{n-1}, \ldots, D_1$, D relative to D_n, and R_{n-1} be the image operator associated with $D_{n-2}, \ldots, D_1$, D relative to D_{n-1}. We wish to establish a relation between R_n and R_{n-1}.

It follows from the definition of these operators that:

$$f_{n-1} = R_{n-1}(\phi + \phi_{n-1}). \tag{49}$$

On the other hand, we have

$$f_n = R_n(\phi). \tag{50}$$

Enforcing the continuity of displacement across $r = a_{n-1}$, we obtain

$$\phi_{n-1} - P_{n-1}(f_{n-1}) = (m_{n-1}^{-1} - 1)\phi - m_n^{-1} f_n, \tag{51}$$

where we define the operators P_j, $j = 1, \ldots, n-1$ on an analytic function g by

$$P_j(g)(z) = g(p_j z). \tag{52}$$

Another relation can be obtained by requiring that the traction be continuous across $r = a_{n-1}$. This yields

$$\phi_{n-1} + P_{n-1}(f_{n-1}) = f_n. \tag{53}$$

Upon adding and subtracting Eqs. (51) and (53), we obtain

$$2\phi_{n-1} = (m_n^{-1} - 1)(\phi - f_n) \tag{54}$$

and

$$2P_{n-1}(f_{n-1}) = (m_n^{-1} + 1) f_n - (m_n^{-1} - 1)\phi. \tag{55}$$

Substituting Eqs. (49) and (50) into the last two relations yields

$$P_{n-1}R_{n-1}\left((m_n^{-1} + 1)\phi - (m_n^{-1} - 1)R_n(\phi)\right) = \left(1 - m_n^{-1}\right)\phi + \left(1 + m_n^{-1}\right)R_n(\phi), \tag{56}$$

which gives

$$\left(m_n^{-1} + 1\right)R_n(\phi) + \left(m_n^{-1} - 1\right)P_{n-1}R_{n-1}R_n(\phi) = \left(m_n^{-1} - 1\right)\phi + \left(m_n^{-1} + 1\right)P_{n-1}R_{n-1}(\phi). \tag{57}$$

Hence

$$(I + \beta_n P_{n-1} R_{n-1}) R_n = \beta_n I + P_{n-1} R_{n-1}, \tag{58}$$

where I is the identity operator and β_n is defined by

$$\beta_n = \frac{m_n^{-1} - 1}{m_n^{-1} + 1} = -\alpha_n. \tag{59}$$

Eq. (58) implies

$$R_n = (I + \beta_n P_{n-1} R_{n-1})^{-1} (\beta_n I + P_{n-1} R_{n-1}). \tag{60}$$

The last equation expresses the image operator of n domains in terms of that of $n-1$ domains, and it is similar to the one derived by Honein et al. (1994) when all singularities are inside the matrix.

To write down the solution to this heterogeneous problem explicitly, let us now assume that the solution to the corresponding homogeneous problem, ϕ, has the following Laurent series expansion in the domain $r > a_{n-1}$

$$\phi = \sum_{k=0}^{\infty} b_k z^{-k}, \tag{61}$$

where b_k, $k = 0, \ldots, \infty$ are complex coefficients, then, following the same line of arguments as in afore-mentioned paper, one can show that the solution to the multilayered problem, when all singularities are inside the inclusion core, is given by

$$f_n = \sum_{k=0}^{\infty} \beta_n * p_{n-1}^{-k} (\beta_{n-1} * \cdots) b_k z^{-k} = \sum_{k=0}^{\infty} \beta_n * q_{n-1}^{k} (\beta_{n-1} * \cdots) b_k z^{-k}, \tag{62}$$

where we recall that the $*$ operation is defined by

$$x * y = \frac{x + y}{1 + xy}. \tag{63}$$

Once f_n is given, all remaining complex potentials can be obtained by induction by enforcing continuity of traction and displacement across the various interfaces.

Eq. (62) provides an explicit solution of the heterogeneous problem with an arbitrary number of layers, singularities being inside the inclusion core, in terms of that of the homogeneous one.

6. The material force on a screw dislocation residing in the inclusion core

In this section, we consider the example of a screw dislocation located inside an inclusion core and interacting with a matrix through an arbitrary number of layers.

Without loss of generality, we assume that the screw dislocation is located at the point c of the x_1- axis where $|c| < a_{n-1}$ (see Fig. 10).

The complex potential of the corresponding homogeneous problem (i.e., when the layers and the matrix are absent and the inclusion-core material occupies the whole space) is given by (see Eq. (5))

$$\phi = \frac{b\mu_n}{2\pi} \ln(z - c), \tag{64}$$

where b is a real constant designating the third component of the Burgers vector.

The complex potential ϕ can be written as

$$\phi = \frac{b\mu_n}{2\pi} \left(\ln\left(1 - \frac{c}{z}\right) + \ln(z) \right). \tag{65}$$

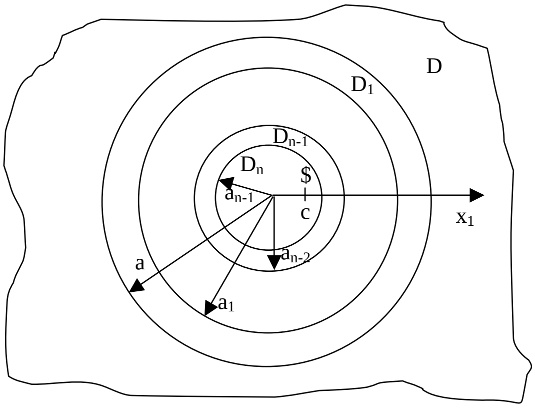

Fig. 10. A screw dislocation in the inclusion core interacting with an arbitrary number of layers.

In the domain $r > a_{n-1}$, the first term is analytic *including the point at infinity* and we have the following expansion:

$$\ln\left(1-\frac{c}{z}\right) = -\sum_{k=1}^{\infty}\frac{c^k}{kz^k}. \tag{66}$$

Thus, it immediately follows from Eq. (62) that:

$$f_n = -\frac{b\mu_n}{2\pi}\sum_{k=1}^{\infty}\beta_n * q_{n-1}^k(\beta_{n-1} * \cdots)\frac{c^k}{kz^k}. \tag{67}$$

The solution inside the inclusion core can now be written as

$$\Phi_n = \frac{b\mu_n}{2\pi}\ln(z-c) + H_{a_{n-1}}(f_n). \tag{68}$$

To evaluate the material (or Peach–Koehler) force acting on the screw dislocation, we apply Budiansky and Rice's formula (25), which leads to

$$J_1 - iJ_2 = b(H(f))'\big|_{z=c}, \tag{69}$$

which, upon using (67), yields

$$J_1 - iJ_2 = -\frac{b^2\mu_n}{2\pi c}\sum_{k=1}^{\infty}\beta_n * q_{n-1}^k(\beta_{n-1} * \cdots)\left(\frac{c}{a_{n-1}}\right)^{2k}. \tag{70}$$

This result yields the material force (or the Peach–Koehler force, in the case of a screw dislocation) acting on the point singularity.

Since $|\beta_n * q_{n-1}^k(\beta_{n-1} * \cdots)| < 1$, the series given by (70) is rapidly convergent; for it has a convergent geometric series as a majorant, and it yields the material force acting on a screw dislocation inside an inclusion core when *an arbitrary number of layers* is present between the inclusion core and the matrix. Furthermore, an upper bound on the error when the series (70) is truncated for the purpose of numerical computation can be easily derived.

For the case $n = 2$ (a homogeneous interphase layer between the inclusion and the matrix,) our result for the complex function f_2 reduces to

$$f_2 = -\frac{b\mu_2}{2\pi}\sum_{k=1}^{\infty}\frac{\beta_2+\beta_1 q_1^k}{1+\beta_2\beta_1 q_1^k}\frac{c^k}{kz^k}, \tag{71}$$

and Eq. (70) takes the form

$$J_1 - iJ_2 = -\frac{b^2\mu_2}{2\pi c}\sum_{k=1}^{\infty}\frac{\beta_2+\beta_1 q_1^k}{1+\beta_2\beta_1 q_1^k}\left(\frac{c}{a_1}\right)^{2k}. \tag{72}$$

This last result is in agreement with the one derived by by Liu et al. (2003), although our equation is written down in a much more compact form.

Eq. (70) is similar to (27), which yields the material force on a dislocation in the matrix, and similar results to those obtained in Section 4 are expected. This is not surprising in light of the involution correspondence that exists in anti-plane elasticity between the solution of a cylinder and that of an infinite domain with a cylindrical hole.

The case of a multi-layered cylinder bonded to a rigid enclosure (zero displacement at the boundary) can be obtained from Eq. (70) by setting

$$\beta_1 = 1. \tag{73}$$

In Fig. 11, we plot the dimensionless material force $F_0 = \frac{2\pi a_1}{\mu_2 b^2}J_1$, as a function of the dimensionless dislocation location given by $\frac{c}{a_1}$, for various values of the shear modulus of a homogeneous single layer coating the inclusion core. As can be seen, for small values of μ_1 (e.g., 0.5), the effect of the soft matrix remains dominant even as the dislocation approaches the boundary. However as the value of μ_1 increases the layer starts asserting itself and the force which is attractive away from the boundary becomes repulsive as the

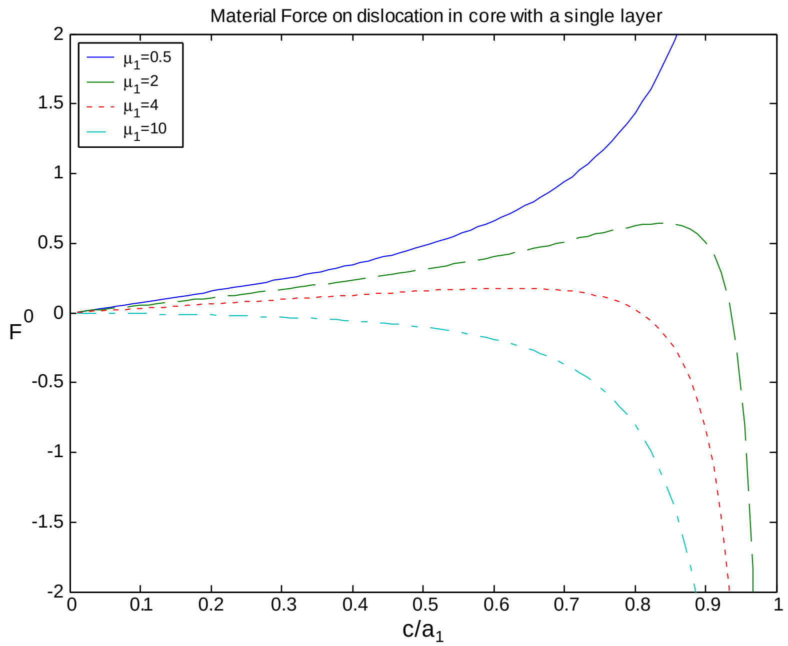

Fig. 11. The material force acting on a screw dislocation in the inclusion core matrix due to the presence of a single homogeneous layer bonded to a soft matrix for various values of the shear modulus μ_1. The parameters are $\mu_0 = 0.1$, $\mu_2 = 1$, $a_0 = 1$ and $a_1 = 0.9$.

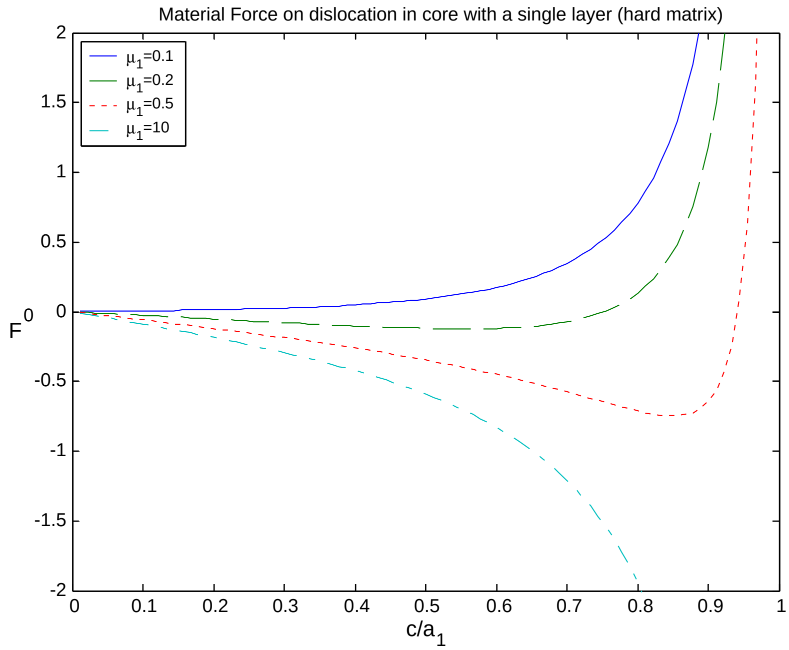

Fig. 12. The material force acting on a screw dislocation in the inclusion core matrix due to the presence of a single homogeneous layer bonded to a hard matrix for various values of the shear modulus μ_1. The parameters are $\mu_0 = 20$, $\mu_2 = 1$, $a_0 = 1$ and $a_1 = 0.9$.

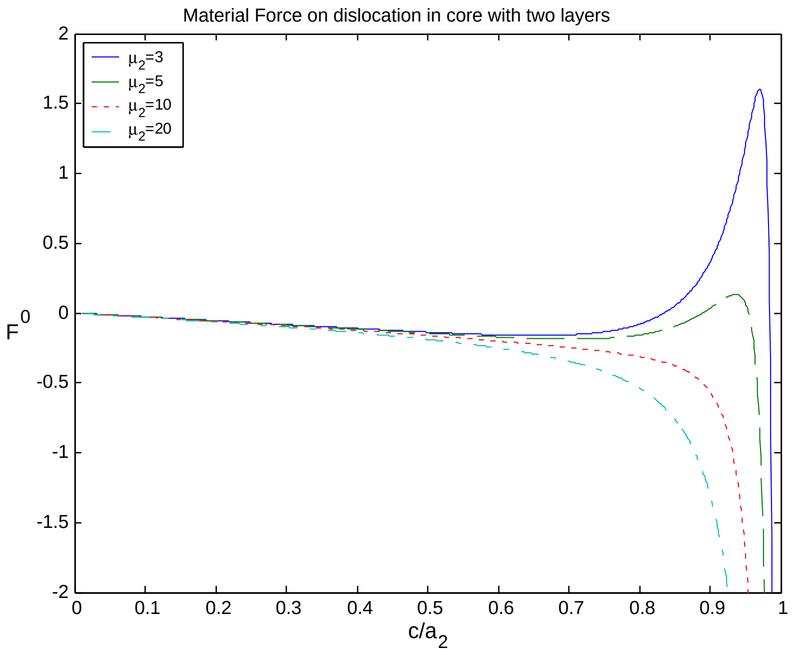

Fig. 13. The material force acting on a screw dislocation in the inclusion core due to the presence of a hard matrix and two homogeneous layers for various values of the shear modulus μ_2. The other parameters are $\mu_0 = 5$, $\mu_1 = 0.08$, $\mu_3 = 1$, $a_0 = 1$, $a_1 = 0.97$ and $a_2 = 0.96$.

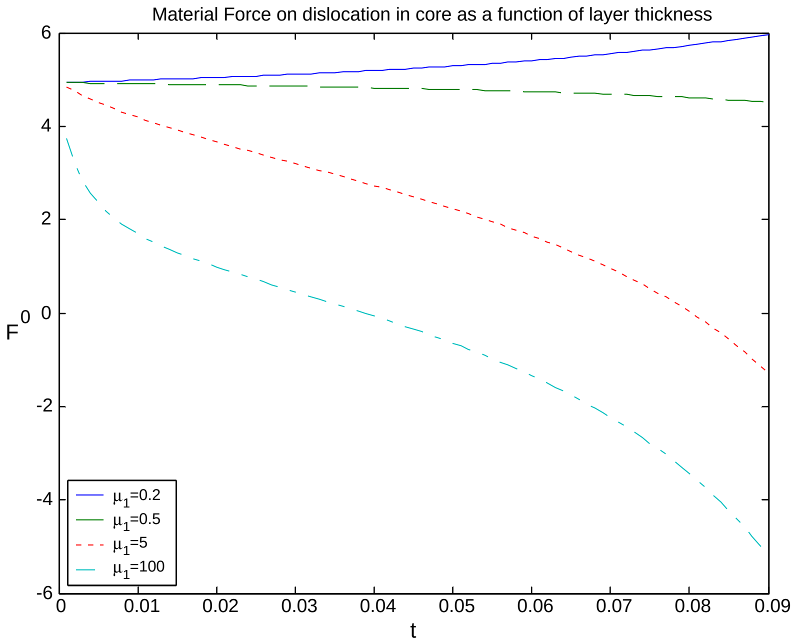

Fig. 14. The material force, plotted as a function of layer thickness, acting on a screw dislocation in the inclusion core due to the presence of a hard matrix and two homogeneous layers for various values of the shear modulus μ_1. The parameters are μ_0=0.1, $\mu_2 = 0.4$, $\mu_3 = 1$, $a_0 = 1$, $a_1 = a_0 - t$, $a_2 = 0.9$ and $c = 0.85$.

dislocation nears the layer, thus giving rise to an equilibrium position, which can be shown to be unstable. As the value of the shear modulus increases further the layer becomes dominant and the force becomes repulsive throughout the range and decays rapidly to zero as the dislocation moves toward the center.

When the soft matrix is replaced with a hard one, the sign of the force is reversed as is shown in Fig. 12 and the unstable equilibrium positions become stable.

As shown in Fig. 13, the presence of an additional layer will alter the character of the solution and for certain combinations of the shear moduli, two equilibrium positions are possible. However, only one of them would be stable. Also, this figure shows that for a given thickness, any of the layers may become dominant if its shear modulus is high enough.

Fig. 14 shows the variation of the material force, on a screw dislocation with a fixed location, as a function of the thickness of one of the layers. Typically, the layer becomes dominant as its thickness increases and the figure shows the typical decrease in the force magnitude with the increase in thickness. It is interesting to note, however, that this assertion does not always hold true as is seen for the case $\mu_1 = 0.2$.

7. Conclusions

In this paper, we presented a novel solution for a multi-layered media in anti-plane elastostatics when all singularities are inside the inclusion core. The essence of the new methodology is a recursive scheme which yields the solution to this heterogeneous problem as a transformation performed on the known solution to the corresponding homogeneous problem. This transformation can be expressed concisely and elegantly in terms of a group structure on the set $I = (-1, 1)$, of real numbers, x, such that $-1 < x < 1$. This work extends a prior research by Honein et al. (1994) who considered the case where the singularities are in the matrix. The solution given here is an *explicit* rapidly convergent Laurent series, whose terms are obtained by multiplying those of the corresponding homogeneous problem with certain appropriate coefficients, which depend only upon the radii of the various layers and their shear moduli. The number of the concentric layers may be arbitrary as well their thicknesses and shear moduli. The singularities are also arbitrary. The two special cases of a screw dislocation either inside the matrix or in the inclusion core were dealt with. The Peach–Koehler force was calculated explicitly as a rapidly convergent series and numerous examples illustrating the effect of the layers have been given.

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